APEC3001 **Spring 2021**

Problem Set 4 – Answer Key

Due Date: Thursday March 18 by 10PM

Please submit one file via Canvas with the questions answered in order.

1. Abel, Baker, and Charlie all run competing bakeries, where each makes loaves of bread.

    a. At Abel's bakery, the marginal product of labor is 15 and the average product of labor is 12. Would Abel's average product increase or decrease if he hired another worker?
    Solution:
    Average product would increase, because marginal product exceeds average product at this quantity.

    b. At Baker's bakery, the marginal product of labor is 7 and the average product of labor is 12. Would Baker's average product increase or decrease if he hired another worker?
    Solution:
    Average product would decrease, because marginal product is below average product at this quantity.

    c. At Charlie's bakery, the MP_L is -12. Does this mean her average product must also be negative?
    Solution:
    No. A negative marginal product means that average product is declining. Yet average product can still be positive.

2. Miguel and Jake run a paper company. Each week they need to produce 1,000 reams of paper to ship to their customers. The paper plant's long-run production function is $Q = 4K^{0.75}L^{0.25}$, where Q is the number of reams produced, K is the quantity of capital rented, and L is the quantity of labor hired. For this production function, the $MP_L = K^{0.75}/L^{0.75}$ and the $MP_K = 3L^{0.25}/K^{0.25}$. The weekly cost function for the paper plant is $C = 10K + 2L$, where C is the total weekly cost.

    a. What ratio of capital to labor minimizes Miguel and Jake's total costs?
    Solution:
    Using the optimality condition, the ratio of capital to labor that minimizes their total costs is
    $$-\frac{W}{R} = -\frac{MP_L}{MP_K}$$
    so that
    $$\frac{2}{10} = \frac{K}{3L}$$
    which implies that
    $$\frac{K}{L} = \frac{3}{5} = 0.6$$
    Therefore, when costs are being minimized, the firm will rent 0.6 units of capital for every unit of labor they hire (Or rent 3 units of capital for every 5 units of labor).

b. (Approximately) how much capital and labor will Miguel and Jake need to rent and hire in order to produce 1,000 reams of paper each week?
Solution:
Since $K = 0.6L$, the quantity of labor employed is

$$Q = 4K^{0.75}L^{0.25}$$
$$1{,}000 = 4 \times (0.6 \times L)^{0.75}L^{0.25}$$
$$250 = (0.6)^{0.75}L$$
$$L \approx 367$$
$$K \approx 0.6 \times 367$$
$$K \approx 220$$

Thus, to produce 1,000 reams of paper, Miguel and Jake need to rent approximately 220 units of capital and hire approximately 367 units of labor.

c. How much will hiring these inputs cost them?
Solution:
The total cost of production is

$$\mathbf{10K + 2L \approx \$10 \times 220 + \$2 \times 367 = \$2{,}934}$$

3. For the following production functions,
   - Find the marginal product of each input.
   - Determine whether the production function exhibits diminishing marginal returns to each input.
   - Find the marginal rate of technical substitution.

a. *Q(K, L) = 3K + 2L*
Solution:
Take the partial derivatives with respect to each input to find the marginal products:

$$\boldsymbol{MP_K = \frac{\partial Q}{\partial K} = 3}$$
$$\boldsymbol{MP_L = \frac{\partial Q}{\partial L} = 2}$$

Marginal products are constant.
Divide the marginal products to get $MRTS_{LK}$:

$$\boldsymbol{MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{2}{3}}$$

b. *Q(K, L) = $K^{0.25}L^{0.5}$*
Solution:
Take the partial derivatives with respect to each input to find the marginal products:

$$\boldsymbol{MP_K = \frac{\partial Q}{\partial K} = 0.25K^{-0.75}L^{0.5}}$$
$$\boldsymbol{MP_L = \frac{\partial Q}{\partial L} = 0.5K^{0.25}L^{-0.5}}$$

Both marginal products decrease as the level of its input increases, holding the level of the other input constant.
Divide the marginal products to get $MRTS_{LK}$:

$$\boldsymbol{MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{0.5K^{0.25}L^{-0.5}}{0.25K^{-0.75}L^{0.5}} = \frac{2K}{L}}$$

4. Catalina Films produces video shorts using digital editing equipment (*K*) and editors (*L*). The firm has the production function $Q = 30K^{0.67}L^{0.33}$, where *Q* is the hours of edited footage. The wage is $25, and the rental rate of capital is $50. The firm wants to produce 3,000 units of output at the lowest possible cost.

   a. Write out the firm's constrained optimization problem.
   Solution:

$$\min_{K,L} \mathbf{50K + 25L}\ \ \boldsymbol{s.t.}\ \mathbf{3,000 = 30K^{0.67}L^{0.33}}$$

   b. Write the cost-minimization problem as a Lagrangian.
   Solution:

$$\min_{K,L,\lambda} \mathbf{50K + 25L + \lambda(3,000 - 30K^{0.67}L^{0.33})}$$

   c. Use the Lagrangian to find the cost-minimizing quantities of capital and labor used to produce 3,000 units of output.
   Solution:
   Take the first-order conditions (FOCs):

$$\frac{\partial \mathcal{L}}{\partial K} = 50 - \lambda(20K^{-0.33}L^{0.33}) = 0$$
$$\frac{\partial \mathcal{L}}{\partial L} = 25 - \lambda(10K^{0.67}L^{-0.67}) = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = 3{,}000 - 30K^{0.67}L^{0.33} = 0$$

   Solve for K as a function of L using the first two conditions:

$$\lambda = \frac{50}{20K^{-0.33}L^{0.33}} = \frac{25}{10K^{0.67}L^{-0.67}}$$
$$50(10K^{0.67}L^{-0.67}) = 25(20K^{-0.33}L^{0.33})$$
$$500K^{0.67}K^{0.33} = 500L^{0.33}L^{0.67}$$
$$K = L$$

   Substitute for K in the production constraint and solve for L*:

$$3{,}000 = 30K^{0.67}L^{0.33} = 30L^{0.67}L^{0.33} = 30L$$
$$L^* = 100$$
$$K^* = 100$$

   The cost-minimizing quantities of capital and labor used to produce 3,000 hours of edited footage are 100 editors and 100 units of digital editing equipment.

   d. What is the total cost of producing 3,000 units?
   Solution:

$$\boldsymbol{TC = RK^* + WL^* = 50(100) + 25(100) = \$7,500}$$

   e. How will total cost change if the firm produces an additional unit of output?
   Solution:
   Solve for λ using the optimal bundle of inputs:

$$\lambda = \frac{50}{20K^{-0.33}L^{0.33}} = \frac{50}{20(100)^{-0.33}(100)^{0.33}} = 2.5$$

   So, the cost of producing one more unit of output is $2.50, which is how much total cost will increase to produce 3,001 hours of edited footage.

5. A firm is employing 100 workers (*W* = \$10/hour) and 50 units of capital (*R* = \$20/hour). At these levels, the marginal product of labor is 5 and the marginal product of capital is 10.

   a. Is this firm minimizing costs? If not, what changes should they make? Explain.
   Solution:
   The cost-minimizing condition is:
   $$-\frac{W}{R} = -\frac{MP_L}{MP_K}$$
   The input-price ratio is:
   $$-\frac{W}{R} = \frac{10}{20} = \frac{1}{2}$$
   The marginal rate of technical substitution is:
   $$-\frac{MP_L}{MP_K} = \frac{5}{10} = \frac{1}{2}$$
   Since the input-price ratio is equal to the marginal rate of technical substitution at the given input prices and marginal products, the firm is minimizing costs.

   b. If the price of labor falls to \$5, what changes, if any, should the firm make in the short run? In the long run?
   Solution:
   Rewriting the cost-minimizing condition yields:
   $$\frac{MP_K}{R} = \frac{MP_L}{W}$$
   If W = \$5/hour, we have:
   $$\frac{10}{20} < \frac{5}{5}$$
   Or:
   $$\frac{1}{2} < 1$$
   So,
   $$\frac{MP_K}{R} < \frac{MP_L}{W}$$
   And the firm is not minimizing its cost with these capital and labor input levels.

   In the short run, the firm should increase the amount of labor it hires to decrease the ratio of the marginal product of labor to the wage rate until it equals ½. In the long run, the firm could both increase the amount of labor it hires to decrease the ratio of the marginal product of labor to the wage rate and decrease the amount of capital it rents to increase the ratio of the marginal product of capital to the capital rental rate, until these ratios are equal, and the cost-minimizing condition is met.

6. Maria produces burritos in her small restaurant using the production function $Q = 10K^{0.5}L^{0.5}$, where K is the number of units of capital she rents and L is the number of hours of labor she employs. Capital costs \$10 per unit and labor costs \$5 per hour. She is currently using 16 units of capital.

   a. In the short run how much labor does Maria need to produce 80 burritos?

Solution:
The production function in terms of L as a function of Q & K is:

$$L^{0.5} = \frac{Q}{10K^{0.5}}$$
$$L = \frac{Q^2}{100K}$$

At Q=80 and K=16, we have:

$$L = \frac{(80)^2}{100(16)} = 4$$

In the short run, Maria needs to hire 4 hours of labor to produce 80 burritos

b. How much does it cost to produce 80 burritos in the short run?
Solution:

$$\boldsymbol{C = RK + WL = 10 \times 16 + 5 \times 4 = \$180}$$

c. What is the (long run) **marginal rate of technical substitution** for this production function? Explain how the combination of inputs changes as Maria moves along an isoquant.
Solution:
The marginal product of labor is:

$$MP_L = \frac{\partial Q}{\partial L} = 5K^{0.5}L^{-0.5}$$

The marginal product of capital is:

$$MP_K = \frac{\partial Q}{\partial K} = 5K^{-0.5}L^{0.5}$$

The marginal rate of technical substitution is:

$$\boldsymbol{-\frac{MP_L}{MP_K} = \frac{5K^{0.5}L^{-0.5}}{5K^{-0.5}L^{0.5}} = \frac{K}{L}}$$

As Maria hires more labor hours, moving right along the isoquant, the units of capital she rents will decrease in the long run, but at a decreasing rate. This is because the more labor she hires, the less willing she is to substitute capital for labor to produce the same units of burritos.

d. Approximately what does it cost Maria to produce 80 burritos in the long run? (Hint: Start by finding the cost-minimizing ratio of capital and labor.)
Solution:
The cost-minimizing condition yields:

$$-\frac{W}{R} = -\frac{MP_L}{MP_K}$$
$$\frac{5}{10} = \frac{5K^{0.5}L^{-0.5}}{5K^{-0.5}L^{0.5}}$$
$$\frac{1}{2} = \frac{K}{L}$$
$$L = 2K$$

Substitute this for L in the production function:

$$80 = 10K^{0.5}L^{0.5} = 10K^{0.5}(2K)^{0.5} = 10 \times 2^{0.5}K$$

Solve for K*:

$$K^* = \frac{80}{10 \times 2^{0.5}} \approx 5.66$$

Solve for L*:

$$L^{*} = 2\left(\frac{80}{10 \times 2^{0.5}}\right) \approx 11.32$$

Total cost to produce 80 burritos in the long-run:

$$\boldsymbol{C = RK^{*} + WL^{*} = 10 \times \left(\frac{80}{10 \times 2^{0.5}}\right) + 5 \times \left(\frac{160}{10 \times 2^{0.5}}\right) \approx \$113.14}$$

e. Does this production function exhibit increasing, decreasing or constant returns to scale? Explain how you know.

Solution:

$$Q = 10K^{0.5}L^{0.5}$$
$$q_1 = Q(1,1) = 10(1)^{0.5}(1)^{0.5} = 10$$
$$q_2 = Q(2,2) = 10(2)^{0.5}(2)^{0.5} = 20$$
$$2q_1 = 20$$
$$q_2 = 2q_1$$

This production function exhibits constant returns to scale because if Maria doubles all inputs, she exactly doubles output.