APEC 3001 Discussion

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Today's Agenda

1. Housekeeping
2. Cost Minimization Using the Lagrangian (15 minutes)
3. Cost Minimization Problem Exercise (15 minutes)
4. Returns to Scale (15 minutes)
5. Questions (5 minutes)

Housekeeping

- State your presence in the Zoom chat for a record of attendance
- Take a minute to download these slides from Canvas under Week 8
- Problem Set 4 is due Thursday, March 18th @ 10PM
- Follow link in TA bio on course Canvas page to sign up for Wednesday office hours

Cost Minimization Using the Lagrangian

- You learned how to solve the producer's **cost minimization problem** (**CMP**) by setting the **marginal rate of technical substitution** between labor (L) and capital (K), or $MRTS_{LK}$, equal to the **input price ratio** ($\frac{W}{R}$)
- Here wages (W) is the price of labor and the capital rental rate (R) is the price of capital
- Remember you derive the $MRTS_{LK}$ by calculating the ratio of the marginal product of labor (MP_L) and the marginal product of capital (MP_K), so $MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{W}{R}$
- So this is analogous to $MRS_{XY} = \frac{MU_X}{MU_Y} = \frac{P_X}{P_Y}$ in the consumer's utility maximization problem (UMP)
- We can also solve the CMP by using the **Lagrangian** as we did with the UMP

Reminder: Constrained Optimization Problem

- Let the **objective function** be given by: $f(x, y)$
- Let the **constraint** be given by: $g(x, y) = 0$
- Then the **Lagrangian equation** is: $\mathscr{L}(x, y, \lambda) = f(x, y) + \lambda[g(x, y)]$
- We optimize the Lagrangian equation using **first-order conditions** (**FOCs**) derived from calculating the partial derivatives with respect to x, y, and λ, and setting them equal to zero:

$$\frac{\partial \mathscr{L}}{\partial x} = \frac{\partial f(x, y)}{\partial x} + \lambda \frac{\partial g(x, y)}{\partial x} = 0$$

$$\frac{\partial \mathscr{L}}{\partial y} = \frac{\partial f(x, y)}{\partial y} + \lambda \frac{\partial g(x, y)}{\partial y} = 0$$

$$\frac{\partial \mathscr{L}}{\partial \lambda} = g(x, y) = 0$$

Producer Cost Minimization Problem - Setup

- The producer chooses the combination of K and L that minimizes, $C = RK + WL$, subject to a production function, $\bar{Q} = AK^{\alpha}L^{1-\alpha}$
- We write the CMP as

$$\min_{K,L} C = RK + WL \; s.t. \; \bar{Q} = AK^{\alpha}L^{1-\alpha}$$

- We can rewrite the CMP in Lagrangian form:

$$\min_{K,L} \mathscr{L}(K, L, \lambda) = RK + WL + \lambda(\bar{Q} - AK^{\alpha}L^{1-\alpha})$$

- Take FOCs:

$$\frac{\partial \mathscr{L}}{\partial K} = R - \lambda(\alpha AK^{\alpha-1}L^{1-\alpha}) = 0$$
$$\frac{\partial \mathscr{L}}{\partial L} = W - \lambda[(1-\alpha)AK^{\alpha}L^{-\alpha}] = 0$$
$$\frac{\partial \mathscr{L}}{\partial \lambda} = \bar{Q} - AK^{\alpha}L^{1-\alpha} = 0$$

Solving the Producer Cost Minimization Problem (I)

- First, solve the first two FOCs for λ:

$$\lambda = \frac{R}{\alpha A K^{\alpha-1} L^{1-\alpha}}$$
$$\lambda = \frac{W}{(1-\alpha) A K^{\alpha} L^{-\alpha}}$$

- The denominators represent the marginal product of K (MP_K) and marginal product of L (MP_L), respectively, which we can rearrange to get the cost-minimization condition in slide 4
- λ, in the producer CMP, represents the extra cost of producing an additional unit of output when the firm is minimizing its costs
- You should think of λ as the marginal cost of production

Solving the Producer Cost Minimization Problem (II)

- Next, set the two equations equal to each other:

$$\frac{R}{\alpha A K^{\alpha-1} L^{1-\alpha}} = \lambda = \frac{W}{(1-\alpha) A K^{\alpha} L^{-\alpha}}$$

- We could rearrange this equation to get $\frac{MP_L}{MP_K} = MRTS_{LK} = \frac{W}{R}$, the cost-minimization condition
- Thus, find (K^*, L^*) by solving the equation above for K as a function of L:

$$K = \left[\frac{\alpha}{(1-\alpha)} \frac{W}{R} \right] L$$

- Substitute the RHS of this equation for K in the third FOC (a.k.a the production function):

$$\bar{Q} = A K^{\alpha} L^{1-\alpha}$$

Solving the Producer Cost Minimization Problem (III)

$$\bar{Q} = A\left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}L\right]^{\alpha} L^{1-\alpha}$$

- Next, solve the equation for L to get L^*:

$$L^* = \left[\frac{(1-\alpha)}{\alpha}\frac{R}{W}\right]^{\alpha} \frac{\bar{Q}}{A}$$

- Finally, substitute L^* for L in the equation for K to get K^*:

$$K^* = \left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}\right]^{1-\alpha} \frac{\bar{Q}}{A}$$

- Therefore, the producer's optimal levels of capital and labor are:

$$(K^*, L^*) = \left(\left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}\right]^{1-\alpha} \frac{\bar{Q}}{A}, \left[\frac{(1-\alpha)}{\alpha}\frac{R}{W}\right]^{\alpha} \frac{\bar{Q}}{A}\right)$$

Cost Minimization Problem Exercise - Figure It Out 6A.1

A firm has the production function

$$Q = 20K^{0.2}L^{0.8}$$

where Q is the level of output, K is the number of machine hours, and L is the number of labor hours. Let $R = \$15$, $W = \$10$, and $\bar{Q} = 40,000$ *units*

1. Construct the cost-minimization statement for the firm
2. Set up the firm's CMP in Lagrangian form
3. Find the first-order conditions (FOCs)
4. Find the firm's optimal levels of K and L

Returns to Scale

Refresher on definition:

- **Returns to Scale**: A change in the amount of output in response to a proportional increase in all of the inputs

Three types of returns to scale:

- **Constant Returns to Scale (CRS)**: A production function for which changing all inputs by the same proportion changes the quantity of output by the same proportion
- **Increasing Returns to Scale (IRS)**: A production function for which changing all inputs by the same proportion changes output more than proportionately
- **Decreasing Returns to Scale (DRS)**: A production function for which changing all inputs by the same proportion changes output less than proportionately

Returns to Scale Cont'd.

- How do we know if a production function exhibits CRS, IRS, or DRS?
- A simple approach is to double all inputs in the production function, and determine if the resulting output exactly doubles, more than doubles, or less than doubles the original output
- If we multiply K and L in $Q = f(K, L)$ by 2, such that $q_1 = f(K_1, L_1)$ & $q_2 = f(2K_1, 2L_1)$, and $q_2 = 2q_1$, then $Q = f(K, L)$ exhibits CRS
- For $q_1 = f(K_1, L_1)$ & $q_2 = f(2K_1, 2L_1)$, if $q_2 > 2q_1$, then $Q = f(K, L)$ exhibits IRS
- For $q_1 = f(K_1, L_1)$ & $q_2 = f(2K_1, 2L_1)$, if $q_2 < 2q_1$, then $Q = f(K, L)$ exhibits DRS
- A special case with a Cobb-Douglas production function, $Q = K^a L^b$, allows you to add up the exponents a and b to determine if the function exhibits CRS ($a + b = 1$), IRS ($a + b > 1$), or DRS ($a + b < 1$)

Returns to Scale - Figure It Out 6.4

For each of the following production functions, determine if they exhibit constant, decreasing, or increasing returns to scale

1. $Q = 2K + 15L$
2. $Q = min(3K, 4L)$
3. $Q = 15K^{0.5}L^{0.4}$

Questions

Any remaining questions?

Additional Support Resources

- Boynton Mental Health Services
- Student Counseling Services
- Let's Talk
- Educational Workshops
- Academic Skills Coaching