6.3. The first task is to show that if $\partial q_i^h/\partial x^h$ is independent of (i.e. not a function of) x^h, then $\partial q_i^h/\partial u^h$ is independent of u. Note that in the first derivative, q_i^h is a Marshallian demand function, while in the second derivative it is a Hicksian demand function. Intuitively, this is quite reasonable. If $\partial q_i^h/\partial x^h$ is independent of x^h than it is a function of prices only, not of prices and x^h. But if the change in demand as x^h changes is not a function of x^h, then it is also not a function of u, since an increase in x^h implies an increase in u. More formally, this can be shown by noting that the Hicksian and Marshallian demand functions must be equal to each other:

$$q_i(x^h, \mathbf{p}) = q_i(u^h, \mathbf{p}) = q_i(v(x^h, \mathbf{p}), \mathbf{p})$$

where $v(x^h, \mathbf{p})$ is the indirect utility function. Differentiate both sides by x^h:

$$\frac{\partial q_i(x^h, \mathbf{p})}{\partial x^h} = f(\mathbf{p}) = \frac{\partial q_i(u^h, \mathbf{p})}{\partial u^h} \times \frac{\partial v(x^h, \mathbf{p})}{\partial x^h}$$

where $f(\mathbf{p})$ indicates that $\partial q_i^h/\partial x^h$ is independent of (i.e. not a function of) x^h, so it is a function only of $\mathbf{p}$. Note that any monotonic transformation of the utility function with respect to x, conditional on $\mathbf{p}$, does not affect demand. It is convenient to select a transformation that gives $\partial v(x^h, \mathbf{p})/\partial x^h = 1$, so that that term drops out of the above equation. Then, differentiation of the middle term, $f(\mathbf{p})$, and of the third term in the above equation, with respect to u gives:

$$0 = \frac{\partial^2 q_i(u^h, \mathbf{p})}{\partial (u^h)^2}$$

which implies that $\partial q_i^h/\partial u^h$ is independent of u. [**There may be another way to show this that is more rigorous.]**

Next, show that $\partial[\partial c^h/\partial u^h]/\partial p_i$ is independent of u^h. First, recall that for any function with continuous derivatives the order of differentiation does not matter. Thus $\partial[\partial c^h/\partial u^h]/\partial p_i = \partial[\partial c^h/\partial p_i]/\partial u^h$. Recall also from Shephard's lemma that $\partial c^h/\partial p_i = q_i(u^h, \mathbf{p})$. Thus $\partial[\partial c^h/\partial p_i]/\partial u^h = \partial q_i(u^h, \mathbf{p})/\partial u^h$. We showed above that $\partial q_i(u^h, \mathbf{p})/\partial u^h$ is independent of u^h, thus $\partial[\partial c^h/\partial p_i]/\partial u^h$, which also equals $\partial[\partial c^h/\partial u^h]/\partial p_i$, is independent of u^h.

Finally, show that equation (1.6) in Chapter 6 can be derived from equation (1.4), where $b(\mathbf{p})$ in (1.6) equals $\partial c^h/\partial u^h$ (and explain why $b(\mathbf{p}) = \partial c^h/\partial u^h$ must be independent of h).

To answer the question in parentheses, note that:

$$\frac{\partial q_i(u^h, \mathbf{p})}{\partial u^h} = \frac{\partial q_i(C(u^h, \mathbf{p}), \mathbf{p})}{\partial u^h} = \frac{\partial q_i(x^h, \mathbf{p})}{\partial x^h} \frac{\partial C(u^h, \mathbf{p})}{\partial u^h}$$

which implies that $\frac{\partial C(u^h, \mathbf{p})}{\partial u^h} = \frac{\partial q_i(u^h, \mathbf{p}) / \partial u^h}{\partial q_i(x^h, \mathbf{p}) / \partial x^h}$

we saw above that $\partial q_i/\partial u^h$ and $\partial q_i/\partial x^h$ are both functions of prices only, and so their ratio must also be a function of prices only, and so $\partial c(u^h, \mathbf{p})/\partial u^h$ must also be a function of prices only, and so not of a function of u^h.

To derive the cost function, that is equation (1.6), take the functional form for the demand function, $q_i^h = \alpha_i^h(\mathbf{p}) + \beta_i(\mathbf{p})x^h$ and note that we can replace x^h with $c(u^h, \mathbf{p})$. Rearranging this gives:

$$c(u^h, \mathbf{p}) = [q_i^h(u^h, \mathbf{p}) - \alpha_i^h(\mathbf{p})]/\beta_i(\mathbf{p})$$

We know from the first part of this problem that $\partial q_i^h(u^h, \mathbf{p})/\partial u^h = f(\mathbf{p})$, that is, it is not a function of u but only a function of $\mathbf{p}$. This implies that $q_i^h(u^h, \mathbf{p})$ must take the form:

$$q_i^h(u, \mathbf{p}) = \gamma_i^h(\mathbf{p}) + \delta_i(\mathbf{p})u^h$$

for some functions $\gamma_i^h(\mathbf{p})$ and $\delta_i(\mathbf{p})$. (This can also be seen by integrating $\partial q_i^h(u^h, \mathbf{p})/\partial u^h$, which conditional on $\mathbf{p}$ is a constant, with respect to u^h, which will lead to that constant multiplied by u^h, plus an undetermined constant which could vary over households.) Substituting this into the above expression implies:

$$c(u^h, \mathbf{p}) = [\gamma_i^h(\mathbf{p}) + \delta_i(\mathbf{p})u^h - \alpha_i^h(\mathbf{p})]/\beta_i(\mathbf{p})$$

$$= \{[\gamma_i^h(\mathbf{p}) - \alpha_i^h(\mathbf{p})]/\beta_i(\mathbf{p})\} + [\delta_i(\mathbf{p})/\beta_i(\mathbf{p})]u^h$$

Thus we can define $a^h(\mathbf{p})$ in (1.6) as $[\gamma_i^h(\mathbf{p}) - \alpha_i^h(\mathbf{p})]/\beta_i(\mathbf{p})$ and $b(\mathbf{p})$ in (1.6) as $\delta_i(\mathbf{p})/\beta_i(\mathbf{p})$. This proves that (1.4) implies (1.6).