APEC 8001: Problem Set 3

1) Let $u(x) = \alpha(x)$

Show that $\alpha(x) \geq \alpha(y) \Rightarrow x \succsim y$

Given $\alpha(x) \geq \alpha(y)$, and $\alpha e \in Z$, where Z, e, and α are defined as in the lecture notes:

- monotonicity implies $\alpha(x)e \succsim \alpha(y)e$
- we also have:
  - $\bar{\alpha}e \gg x \Rightarrow \bar{\alpha}e \succsim x$ by monotonicity
  - $\bar{\alpha}e \gg y \Rightarrow \bar{\alpha}e \succsim y$ by monotonicity
- $\exists \alpha(x) \in [0, \bar{\alpha}]$ such that $\alpha(x)e \sim x$ by continuity and monotonicity
- Similarly, $\exists \alpha(y) \in [0, \bar{\alpha}]$ such that $\alpha(y)e \sim y$
- $\alpha(x) \geq \alpha(y) \Rightarrow \alpha(x)e \gg \alpha(y)e$, and so monotonicity implies $\alpha(x)e \succsim \alpha(y)e$
- By transitivity, $x \sim \alpha(x)e \succsim \alpha(y)e \sim y \Rightarrow x \succsim y$
  - to see this, show $\forall x, y, z \in X$, $x \sim y$ and $y \succsim z \Rightarrow x \succsim z$
    - $x \sim y \Rightarrow x \succsim y$ and $y \succsim x$ by definition of $\sim$
    - transitivity of $\succsim$ yields $x \succsim y \succsim z \Rightarrow x \succsim z$
  - so $x \sim \alpha(x)e$ and $\alpha(x)e \succsim \alpha(y)e \Rightarrow x \succsim \alpha(y)e$
  - Similarly, $x \succsim \alpha(y)e$ and $\alpha(y)e \sim y \Rightarrow x \succsim y$
- Therefore, $\alpha(x) \geq \alpha(y) \Rightarrow x \succsim y$

$\blacksquare$

2a) Gradient of $u(x_1, x_2) = \begin{bmatrix} \frac{1}{2x_1^{1/2}} \\ \frac{1}{2x_2^{1/2}} \end{bmatrix}$

$x_1, x_2 \geq 0 \Rightarrow \frac{1}{2x_1^{1/2}}, \frac{1}{2x_2^{1/2}} > 0,$

because x_1 & $x_2 \neq 0$

$\therefore u(x_1, x_2)$ is increasing in x

Hessian of $u(x_1, x_2) = \begin{bmatrix} -\frac{1}{4x_1^{3/2}} & 0 \\ 0 & -\frac{1}{4x_2^{3/2}} \end{bmatrix}$

Similarly, $-\frac{1}{4x_1^{3/2}}, -\frac{1}{4x_2^{3/2}} < 0$

$\Rightarrow$ Hessian is negative semidefinite

$\therefore u(x_1, x_2)$ is concave in x

2b) Budget constraint: $w = p_1 x_1 + p_2 x_2$

UMP: $\max_{x_1, x_2 \geq 0} x_1^{1/2} + x_2^{1/2}$ subject to $w = p_1 x_1 + p_2 x_2$

2c) $\mathcal{L} = x_1^{1/2} + x_2^{1/2} + \lambda(w - p_1x_1 - p_2x_2)$

FOCs:

① $\frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{2x_1^{1/2}} - \lambda p_1 \leq 0$; $\frac{\partial \mathcal{L}}{\partial x_1} \cdot x_1 = 0$; $x_1 \geq 0$

② $\frac{\partial \mathcal{L}}{\partial x_2} = \frac{1}{2x_2^{1/2}} - \lambda p_2 \leq 0$; $\frac{\partial \mathcal{L}}{\partial x_2} \cdot x_2 = 0$; $x_2 \geq 0$

③ $\frac{\partial \mathcal{L}}{\partial \lambda} = w - p_1x_1 - p_2x_2 = 0$ (By Walras' law)

Interior solution: $x_1, x_2 > 0$
yields:

$$x_1^* = \frac{w p_2}{p_1(p_1+p_2)} \qquad x_2^* = \frac{w p_1}{p_2(p_1+p_2)}$$

Corner solutions:
- $x_1, x_2 = 0$ isn't possible because of monotonicity and $w > 0$
- WLOG, let $x_i = 0$, $x_j > 0$, then $\frac{\partial \mathcal{L}}{\partial x_i}$ is undefined, thus $x_i \neq 0$
- no corner solutions

3a) $\max_{x_1, x_2 \geq 0} (x_1 - \gamma_1)^\beta (x_2 - \gamma_2)^{1-\beta}$ subject to $w = p_1 x_1 + p_2 x_2$

- use lagrangian and FOCs to find:

$$x_1(p, w) = \frac{\beta(w - p_1\gamma_1 - p_2\gamma_2)}{p_1} + \gamma_1$$

$$x_2(p, w) = \frac{(1-\beta)(w - p_1\gamma_1 - p_2\gamma_2)}{p_2} + \gamma_2$$

3b) $\min_{x_1, x_2 \geq 0} p_1 x_1 + p_2 x_2$ subject to $u = (x_1 - \gamma_1)^\beta (x_2 - \gamma_2)^{1-\beta}$

- use lagrangian and FOCs to find:

$$h_1(p, u) = u\left(\frac{\beta\, p_2}{(1-\beta) p_1}\right)^{1-\beta} + \gamma_1$$

$$h_2(p, u) = u\left(\frac{(1-\beta) p_1}{\beta\, p_2}\right)^{\beta} + \gamma_2$$

3c) $$v(p, w) = u(x_1(p, w), x_2(p, w))$$
$$= \left(\frac{\beta(w - p_1\gamma_1 - p_2\gamma_2)}{p_1}\right)^\beta \left(\frac{(1-\beta)(w - p_1\gamma_1 - p_2\gamma_2)}{p_2}\right)^{1-\beta}$$

3d) $$e(p, u) = p_1 h_1(p, u) + p_2 h_2(p, u)$$
$$= u\left(\frac{p_1}{\beta}\right)^\beta \left(\frac{p_2}{1-\beta}\right)^{1-\beta} + p_1\gamma_1 + p_2\gamma_2$$

3e) Roy's Identity: $x_i(p,w) = -\dfrac{\partial v(p,w)/\partial p_i}{\partial v(p,w)/\partial w}$

$$x_1(p,w) = -\frac{-\beta^{\beta}(1-\beta)^{1-\beta} p_1^{-\beta-1} p_2^{\beta-1}[\beta(w-p_1\gamma_1-p_2\gamma_2)-p_1\gamma_1]}{\beta^{\beta}(1-\beta)^{1-\beta} p_1^{-\beta} p_2^{\beta-1}}$$

$$= \frac{\beta(w-p_1\gamma_1-p_2\gamma_2)}{p_1} + \gamma_1$$

$$x_2(p,w) = -\frac{-\beta^{\beta}(1-\beta)^{1-\beta} p_1^{-\beta} p_2^{\beta-2}[(1-\beta)(w-p_1\gamma_1-p_2\gamma_2)-p_2\gamma_2]}{\beta^{\beta}(1-\beta)^{1-\beta} p_1^{-\beta} p_2^{\beta-1}}$$

$$= \frac{(1-\beta)(w-p_1\gamma_1-p_2\gamma_2)}{p_2} + \gamma_2$$

Same demands from part a)

3f) Sheperd's Lemma: $h_i(p,u) = \dfrac{\partial e(p,u)}{\partial p_i}$

$$h_1(p,u) = u\left(\frac{p_1}{\beta}\right)^{\beta-1}\left(\frac{p_2}{1-\beta}\right)^{1-\beta} + \gamma_1$$

$$= u\left(\frac{\beta p_2}{(1-\beta)p_1}\right)^{1-\beta} + \gamma_1$$

$$h_2(p,u) = u\left(\frac{p_1}{\beta}\right)^{\beta}\left(\frac{p_2}{1-\beta}\right)^{-\beta} + \gamma_2$$

$$= u\left(\frac{(1-\beta)p_1}{\beta p_2}\right)^{\beta} + \gamma_2$$

Same demands from part b)

3g) $x_i(p,w) = h_i(p, v(p,w))$

$$x_1(p,w) = v(p,w)\left(\frac{\beta p_2}{(1-\beta)p_1}\right)^{1-\beta} + \gamma_1$$

$$= \left(\frac{\beta(w-p_1\gamma_1-p_2\gamma_2)}{p_1}\right)^{\beta}\left(\frac{(1-\beta)(w-p_1\gamma_1-p_2\gamma_2)}{p_2}\right)^{1-\beta}\left(\frac{\beta p_2}{(1-\beta)p_1}\right)^{1-\beta} + \gamma_1$$

$$= \frac{\beta(w-p_1\gamma_1-p_2\gamma_2)}{p_1} + \gamma_1$$

$$x_2(p,w) = v(p,w)\left(\frac{(1-\beta)p_1}{\beta p_2}\right)^{\beta} + \gamma_2$$

$$= \left(\frac{\beta(w-p_1\gamma_1-p_2\gamma_2)}{p_1}\right)^{\beta}\left(\frac{(1-\beta)(w-p_1\gamma_1-p_2\gamma_2)}{p_2}\right)^{1-\beta}\left(\frac{(1-\beta)p_1}{\beta p_2}\right)^{\beta} + \gamma_2$$

$$= \frac{(1-\beta)(w-p_1\gamma_1-p_2\gamma_2)}{p_2} + \gamma_2$$

Same demands from part a)

3h) $h_i(p,u) = x_i(p, e(p,u))$

$$h_1(p,u) = \frac{\beta(e(p,u)-p_1\gamma_1-p_2\gamma_2)}{p_1} + \gamma_1$$

$$= \frac{\beta\left[u\left(\frac{p_1}{\beta}\right)^{\beta}\left(\frac{p_2}{1-\beta}\right)^{1-\beta} + p_1\gamma_1 + p_2\gamma_2 - p_1\gamma_1 - p_2\gamma_2\right]}{p_1} + \gamma_1$$

$$= u\left(\frac{\beta p_2}{(1-\beta)p_1}\right)^{1-\beta} + \gamma_1$$

3h) $h_2(p,u) = \frac{(1-\beta)(e(p,u) - p_1\gamma_1 - p_2\gamma_2)}{p_2} + \gamma_2$

$$= \frac{(1-\beta)\left[u\left(\frac{p_1}{\beta}\right)^{\beta}\left(\frac{p_2}{1-\beta}\right)^{1-\beta} + p_1\gamma_1 + p_2\gamma_2 - p_1\gamma_1 - p_2\gamma_2\right]}{p_2} + \gamma_2$$

$$= u\left(\frac{(1-\beta)p_1}{\beta p_2}\right)^{\beta} + \gamma_2$$

Same demands from part b)

3i) Set $v(p,w) = u$ and solve for $w = e(p,u)$

$$\left(\frac{\beta(w - p_1\gamma_1 - p_2\gamma_2)}{p_1}\right)^{\beta}\left(\frac{(1-\beta)(w - p_1\gamma_1 - p_2\gamma_2)}{p_2}\right)^{1-\beta} = u$$

$$\Rightarrow w = u\left(\frac{p_1}{\beta}\right)^{\beta}\left(\frac{p_2}{1-\beta}\right)^{1-\beta} + p_1\gamma_1 + p_2\gamma_2$$

$$\Rightarrow e(p,u) = u\left(\frac{p_1}{\beta}\right)^{\beta}\left(\frac{p_2}{1-\beta}\right)^{1-\beta} + p_1\gamma_1 + p_2\gamma_2$$

3j) Set $e(p,u) = w$ and solve for $u = v(p,w)$

$$u\left(\frac{p_1}{\beta}\right)^{\beta}\left(\frac{p_2}{1-\beta}\right)^{1-\beta} + p_1\gamma_1 + p_2\gamma_2 = w$$

$$\Rightarrow u = \left(\frac{\beta(w - p_1\gamma_1 - p_2\gamma_2)}{p_1}\right)^{\beta}\left(\frac{(1-\beta)(w - p_1\gamma_1 - p_2\gamma_2)}{p_2}\right)^{1-\beta}$$

$$\Rightarrow v(p,w) = \left(\frac{\beta(w - p_1\gamma_1 - p_2\gamma_2)}{p_1}\right)^{\beta}\left(\frac{(1-\beta)(w - p_1\gamma_1 - p_2\gamma_2)}{p_2}\right)^{1-\beta}$$

k) $\frac{\partial h_i(p,u)}{\partial p_j} = \frac{\partial x_i(p,w)}{\partial p_j} + \frac{\partial x_i(p,w)}{\partial w} x_j(p,w) \; \forall i,j$

$$\frac{\partial h_1(p,u)}{\partial p_1} = \frac{\beta(p_2\gamma_2 - w)}{p_1^2} + \frac{\beta}{p_1}\left(\frac{\beta(w - p_1\gamma_1 - p_2\gamma_2)}{p_1} + \gamma_1\right)$$
$$= u\left(\frac{\beta - 1}{p_1}\right)\left(\frac{\beta p_2}{(1-\beta)p_1}\right)^{1-\beta}$$

$$h_1(p,u) = \int u\left(\frac{\beta - 1}{p_1}\right)\left(\frac{\beta p_2}{(1-\beta)p_1}\right)^{1-\beta}$$
$$= u\left(\frac{\beta p_2}{(1-\beta)p_1}\right)^{1-\beta} + C$$

if $C = \gamma_1$, we have $h_1(p,u) = u\left(\frac{\beta p_2}{(1-\beta)p_1}\right)^{1-\beta} + \gamma_1$

Similarly, use the same process to show
$h_2(p,u) = u\left(\frac{(1-\beta)p_1}{\beta p_2}\right)^{\beta} + \gamma_2$

l) Because you don't have all the information you need to solve this PDE, I accepted showing the Slusky equation relationship holds for all i,j.